A progressive correction to the circular hydraulic jump scaling

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I. INTRODUCTION

The hydraulic jump has been extensively studied in the field of Fluid Mechanics. It is an intriguing and interesting effect that arise in different geometries and has attracted the regard of many scientists since the first descriptions by Leonardo da Vinci. As there are no publications associated with his work, and his most famous writings are related to bridges and machines, the first experimental studies are credited to Bidone in 1820.1 Since then, considerable research effort has gone into the study of this subject. Its common axisymmetrical form is known as the circular hydraulic jump which can be easily observed at a kitchen sink. It appears when a vertical jet of fluid is directed upon a hard wall and spreads out radially. At a given radius, one observes a sudden increase in the height of the fluid that in average, is rather stable.

The first description of the circular hydraulic jump was written by Lord Rayleigh2 in 1914, who also noticed the undoubted dependency of the process on surface tension at low flows. One of the first theoretical approaches to the circular hydraulic jump was given in the middle of the 20th century,3 taking into account viscous effects. In Ref. 4, the jump length scales as a function of the flux Q, the nozzle diameter a, and the height of the fluid h downstream the shock (see Fig. 1). As observed in different geometries, the hydraulic jump occurs spontaneously, without the aid of any obstacle. The mechanism was then related to the friction between the fluid and the solid plate. The above reasoning speaks of the important role of the kinematic viscosity ν at the jump formation.

In a theoretical and computational study,5 their predictions are compared with experimental data6 involving non-fixed outer heights. Almost at the same time, a model7 allows to predict the jump radius and the flow depth of the jump as functions of viscosity and supercritical velocity v1. A few years after, it is proposed that the hydraulic jump occurs when the boundary layer δ covers the entire supercritical fluid film.8 By means of elementary hydrodynamics, the scaling laws governing the position of the hydraulic jump are investigated and compared with experimental data. The flow in
FIG. 1. Sketch of the experiment. The radii \( r_1, r_0, r, r_0, 0, r_0, 0, r_0, 0, 0 \) and heights \( h_1, h_0, 1, h_0, h_0, 0, h_0, 0, h_0, 0, 0 \) are ordered as in the water case (see text, Eqs. (12) and (15)-(17)). The position of the external wall of height \( d_0 \) is variable.

the boundary layer can be described as a laminar viscous fluid. In a cylindrical geometry, for a small radius, the jet has a mean velocity \( v_1 \sim (2gd)^{1/2} \), then the size of the boundary layer is approximately \( \delta \sim (\nu r)^{1/2}/(2gd)^{1/4} \), where \( r \) is the radial coordinate and \( d \) is the initial nozzle position. Using three different arguments, it is possible to found the dependence on the mass flux, nozzle height, and viscosity, \( r_d \sim Q^{2/3}d^{-1/6}v^{-1/3} \).

Despite this major breakthrough, the previous results depend on the nozzle height or its diameter \( a \), which are corrections of the initial flux. In Ref. 9, the authors find a scaling law that is independent of these parameters, and the circular hydraulic jump is studied using average equations of the shallow-water type. By connecting inner and outer solutions through a shock, an equation for the radius of the jump is obtained

\[
 r_i \sim \left( \frac{Q}{\nu^3 g} \right)^{1/8}.
\]  

This scaling law agrees with experimental data,\textsuperscript{3,8–10} but the coefficients associated are free parameters and the obtained law does not depend on the final fluid depth, surface tension \( \gamma \) or density \( \rho \), in opposition to experimental evidence.\textsuperscript{11,12} In the latter, they increase or decrease \( h \) by changing the position of \( d_0 \) (see Fig. 1), the external height of the wall at the rim. In Ref. 13, the authors study the influence of surface tension and extend the theory of viscous hydraulic jumps\textsuperscript{4} with surface tension terms that improve comparisons with experimental data specifically at small radius jumps, using moderately viscous fluids (between 1 and 20 times more viscous than water). In this letter, we present numerical solutions of the circular hydraulic jump based on the inertial lubrication theory,\textsuperscript{14} and a progressive approach to estimate the jump length is derived using inertial and nonlinear terms in the limit of shallow and viscous fluid beds. The results are accurate at low and high flows and depend on surface tension, density and subcritical depth, matching previous findings.\textsuperscript{9,11,12,15}

II. GENERAL FORMULATION

The hydraulic jump equation with cylindrical symmetry is obtained from Ref. 14. These lubrication type equations have been also applied to other problems involving thin and viscous layers,\textsuperscript{16} in agreement with experimental results. As the circular hydraulic jump is a roughly stable process, we consider stationary states and no dependency on the angular coordinate. The change of variables used in Ref. 14 for the coordinates \((x, z)\), the free surface \( \xi(x, z) \), the velocity field \((v_\perp, v_z)\), the time \( t \), and the pressure \( p \) is

\[
\mathbf{\tilde{x}} = \frac{x}{L}, \quad (\mathbf{\tilde{z}}, \mathbf{\tilde{\xi}}) = \frac{1}{h}(z, \xi), \quad \mathbf{\tilde{v}}_\perp = \frac{1}{\Omega L}(v, v_0),
\]

\[
\mathbf{\tilde{v}}_z = \frac{1}{\Omega h}v_z, \quad \mathbf{\tilde{t}} = \Omega t, \quad \tilde{p} = \frac{h^2}{\rho v \Omega L^2} p,
\]

where the tilde notation corresponds to non-dimensional variables, \( L \) is the radius of the plate, and \( \Omega^{-1} \) is the time scale of the process (see Fig. 1). The continuity equation, related to the mass conservation
of the system is \( \partial_\xi \tilde{\xi} + \tilde{\nabla}_\perp \cdot \tilde{\mathbf{q}} = 0 \), with \( \tilde{\mathbf{q}} = \int_0^\xi \tilde{\nabla}_\perp d\xi \), reduced to \( \tilde{q} = \int_0^\xi \tilde{v}_d d\xi \) (due to the cylindrical symmetry, we consider the angular velocity field \( \mathbf{v}_\theta = 0 \)), which yields \( \frac{1}{2} \frac{d}{dt}(\tilde{r} \tilde{q}) = 0 \), that implies \( \tilde{q}(\tilde{r}) = C/\tilde{r} \) where \( C \) is a constant that will be determined and \( \tilde{\nabla}_\perp \) is the horizontal gradient. As defined in Ref. 9, \( Q = 2\pi r \int_0^\xi \tilde{v}_d d\xi \). Using (2) we have \( Q = 2\pi \Omega L^2 h \tilde{r} \int_0^\xi \tilde{v}_d d\xi = 2\pi \Omega L^2 h \tilde{r} \tilde{q}(\tilde{r}) \). Using the solution \( \tilde{q}(\tilde{r}) = C/\tilde{r} \), we find \( C = Q/2\pi \Omega L^2 h \) as a function of the parameters. From now on we drop the tilde notation (all the variables in (2) are non-dimensional). We define \( A = 2\pi L h \) to be the side area of a fluid cylinder of radius \( L \) and height \( h \) and we have \( C = Q/\Omega AL \).

As shown previously, \( C = q(r) \) is satisfied anywhere on the fluid, with \( q(r) = \tilde{v}_d \xi \) (related to the mean radial velocity \( \tilde{v}_d = \frac{1}{\xi} \int_0^\xi \tilde{v}_d d\xi \) and \( r \) both non-dimensional parameters, using now (2) with dimensional variables we have \( C = \frac{\xi}{h} \frac{r}{L} \xi L = \frac{U}{CL} \), which means that \( r = L \) is the geometric place where \( \xi = h \) and \( \tilde{v}_d = U \). Comparing \( C = Q/\Omega AL = U/\Omega L \) the mean flow velocity is \( U = Q/A \), as expected. Considering \( \Omega^{-1} \) as the length of time for filling the vessel, or the time required to change the entire fluid volume \( V \) in the experiment, \( \Omega^{-1} = V/Q \approx \pi L^2 h / Q \). Using this equation, the Reynolds number is defined as \( Re = \Omega h^2 / \nu \approx Qh/\pi L^2 \). With this definition, the Reynolds number range is between \( Re \approx 10^{-3} \) and \( Re \approx 10 \) according to the experimental data used in this work.11,12,15 Defining the Froude number as \( F = Q^2 h^3 / gh \approx Q^2/\pi L^3 h^2 L^2 \) and using \( q(r) = Clr \), the axisymmetrical deformation of the fluid layer, described in Ref. 14 is rewritten as \( T_3 \xi'' + T_2 \xi'' + T_1 \xi' + T_0 = 0 \), where

\[
T_0 = \frac{3}{2} r^2 - \frac{27}{70} Re \xi, \quad T_1 = r \left( \frac{Re}{F} \xi^3 \left[ r^2 + \frac{e^2}{2} \right] - \frac{e^2}{2} \left[ \frac{24}{5} r \xi' + \frac{27}{10} \xi \right] - \frac{70}{70} Re \right),
\]

\[
T_2 = r^2 e^2 \left( \frac{18}{5} - \frac{Re}{FB_w} \xi^2 \right) \xi, \quad T_3 = -r^3 e^2 \frac{Re}{FB_w},
\]

\( e = h/L \) is the aspect ratio, \( B_w = h^2 / L^2 \) is the Bond number and \( l_c = (\gamma/\rho g)^{1/2} \) is the capillary length that indicates the relative importance of forces induced by surface tension and gravity.

III. RESULTS

Neglecting \( e^2 \), \( Re \) and surface tension terms in (3) we have \( \frac{d\xi}{dr} = -\frac{3F}{2Re \xi^2} \) and its logarithmic solution, where \( \xi_0 \) and \( R_0 \) are initial conditions

\[
\xi(r) = \left( \xi_0^4 - 6 \xi_0^3 \log \left( \frac{r}{R_0} \right) \right)^{1/4},
\]

equivalent to Eqs. (43) and (44) in Ref. 9. This solution vanishes at the outer edge; therefore, it is not a good approximation as explained in the latter reference. Considering the effects of \( Re \) terms in Eqs. (3) we have

\[
\frac{\partial \xi}{\partial r} \left( 1 - \frac{27}{70} \frac{F}{r^2 \xi^3} \right) + \left( \frac{3}{2} \frac{F}{r^2 \xi^3} - \frac{27}{70} \frac{F}{r^3 \xi^2} \right) = 0.
\]

Here, we show how to obtain this equation in the framework given by Ref. 9 and how the proposed free parameters are corrected by the present work. As addressed in that reference, the boundary layer equations in cylindrical coordinates are

\[
\frac{u}{r} + \frac{\partial u}{\partial z} = -g \frac{\partial \xi}{\partial r} + \frac{\partial^2 u}{\partial z^2} \quad \text{and} \quad \frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0.
\]

Applying the change of variables (2), the incompressibility relation (5) (right), is invariant and Eq. (5) (left) yields

\[
Re \left( \frac{u}{r} + \frac{\partial u}{\partial z} \right) = -\frac{Re}{F} \frac{\partial \xi}{\partial r} + \frac{\partial^2 u}{\partial z^2}.
\]
Using the kinematic condition at the surface without explicit time dependence \( w|_{z=\xi} = (v_\perp)|_{z=\xi} \), the following identity is obtained:

\[
\int_0^\xi w \frac{\partial u}{\partial z} \, dz = \frac{\partial \xi}{\partial r} u^2|_{z=\xi} + \int_0^\xi u \left( \frac{\partial u}{\partial r} + \frac{u}{r} \right) \, dz. \tag{7}
\]

Now we calculate the mean gradient of the quadratic radial velocity \( \nabla_r u^2 \) and we use (7) which gives

\[
\nabla_r u^2 = \frac{1}{\xi} \left( \int_0^\xi w \frac{\partial u}{\partial z} \, dz + \int_0^\xi u \frac{\partial u}{\partial r} \, dz \right). \tag{8}
\]

Integrating (6) and using (7) and (8) and the surface boundary condition at the leading order\(^\text{14}\) \( \frac{\partial w}{\partial z}|_{z=\xi} = 0 \) we obtain

\[
Re \nabla_r u^2 = -\frac{Re}{F} \frac{\partial \xi}{\partial r} - \frac{1}{\xi} \frac{\partial u}{\partial z}|_{z=0}. \tag{9}
\]

As proposed in Ref.\(^9\), we assume \( \nabla_r u^2 = C_1 \frac{2}{r} v^2 \) and \( \frac{\partial w}{\partial z}|_{z=0} = C_2(v/\xi) \). Using \( q = C/r \) the average velocity is \( v = C/\xi \), by applying these equations into (9) we obtain

\[
\frac{\partial \xi}{\partial r} \left( 1 - \frac{C_1}{2} \frac{F}{r^2 \xi^3} \right) + \frac{C_2}{2} \frac{1}{Re \xi^3} - \frac{C_1}{2} \frac{F}{r^3 \xi^2} = 0. \tag{10}
\]

Comparing (10) with (4) we identify \( C_1 = 27/35 \) and \( C_2 = 3 \). The first coefficient \( C_1 \) is the correction made in this work to the free parameter\(^8\) \( C_1 = 6/5 \). The second coefficient \( C_2 = 3 \) coincides with the addressed approach.

A. Free surface and velocity field

Numerical solutions of equation (3) show the first forms of the hydraulic jump, obtained from a shooting method using far-field conditions. The imposed boundary conditions depend on the technical scheme and, in general, experimental studies\(^\text{11–13,15,17}\) do not specify in detail the used boundary conditions, for instance, the values of the first and second spatial gradients at the rim. By minimizing the difference between the experimental and numerical solutions (minimum mean square error estimator) we find the first and second derivatives of the surface at the outer edge. Using these values, we obtain numerical solutions using a shooting method for different heights, finding good agreements with experimental results\(^\text{11,12}\) (see Fig. 2).

![FIG. 2. Numerical height profiles \( \xi(r) \) for different values of the external depth for type I jumps (lines). Experimental data (circles) obtained from Ref. 11. Horizontal and vertical scales are normalized with the plate radius \( L = 37.9 \) mm and the lowest final depth \( h = 2.77 \) mm, respectively. Reproduced by permission from Bohr et al., Physica B 228, 1–10 (1996). Copyright 1996 Elsevier B.V.]
We normalize data with the total experimental length \( L = 37.9 \) mm and the lowest supercritical height \( h = 2.77 \) mm. We observe capillary waves produced upstream the jump position that may be reduced by increasing the length of the numerical solutions. We do not consider the curves of type II (as mentioned in Refs. 11 and 12, see Fig. 3) caused by the rim height used in experiments, which often leads to a breaking of the circular symmetry, where it is necessary to consider angular terms.\(^{18-20}\) The velocity field (see Fig. 4) is computed using the definitions provided in Ref. 14 and calculating the time scale of the system \( \Omega^{-1} = V/Q \). In this case, the volume \( V \) occupied by the fluid is calculated by integrating the numerical profile \( \xi(r) \). Here, we extend numerical solutions and normalize experimental data with a radius \( L = 105 \) mm (smaller than the plate radius\(^{11,12}\)) and a final height \( h = 4.1 \) mm.\(^{21}\) The capillary waves are reduced and the surface velocity magnitude matches experimental measures, except just prior to the jump position at the supercritical region. Although increasing the size of the vessel reduces these waves upstream the jump, their presence (specially at the surface velocity, see Fig. 4), is basically related to the truncation of the polynomial expansion in the coordinate \( z \), at the vertical and horizontal velocities, in the derivation of the inertial lubrication theory.\(^{14}\) In the latter, the exact equation that describes free surface flows contains an infinite number of temporal derivatives that we approximate up to the inertial terms.

**B. Scaling laws**

In Eqs. (3), we consider small perturbations \( \delta \xi \) of the free surface \( \xi_j = \epsilon_j + \delta \xi \) and small perturbations \( \delta r \) around the jump radius \( r_j = r_j + \delta r \) (see Fig. 5). Here \( \epsilon_j = h_j/h \), where \( h_j \) is the height of the fluid at the jump position at first order and \( r_j \) is scaled with \( L \) as it is a horizontal variable. Supposing \( \delta \xi \sim \delta r \), terms as \( \delta \xi \sim \delta \xi/\delta r \sim 1 \), \( \delta \xi'' \sim \delta \xi/\delta r^2 \sim \delta r^{-1} \) and so on. This gives a hierarchy of equations. At the highest order \( \mathcal{O}(\delta r^{-2}) \) we obtain \( T(0)_{3,3}^{(0)} \delta \xi'' = 0 \), which implies a quadratic form for the surface at the jump position \( \xi_j = \epsilon_j + c_1 \delta r + c_2 \delta r^2/2 \). The next order \( \mathcal{O}(\delta r^{-1}) \) yields \( T(1)_{2,2}^{(0)} \delta \xi'' = 0 \), then \( \xi_j = \epsilon_j + c_1 \delta r \) and the hypothesis \( \delta \xi \sim \delta r \) is confirmed. At \( \mathcal{O}(\delta r^0) \) the equation is \( T(0)_0^{(0)} + T(0)_1^{(0)} c_1 = 0 \). As \( c_1 \) is unknown, the latter equation is solved each term

![FIG. 3. Schematic figures of type I and type II hydraulic jumps. A second surfing roll is observed when the outer depth is increased.](image1)

![FIG. 4. Horizontal and vertical coordinates \((r, \xi)\) are normalized with \( L = 105 \) mm and \( h = 4.1 \) mm, velocities are normalized using a supercritical speed \( v_1 = 2.14 \) m/s, given by experimental data.\(^{11}\) (Left) Free surface (line) and bulk velocity field (arrows). Dark areas denote low velocities. The velocity field shows a separation eddy causing the hydraulic jump. (Right) Surface velocity (line) compared with experimental data upstream (circles) and downstream (squares) the jump.\(^{11}\) Reproduced by permission from Bohr et al., Physica B 228, 1–10 (1996). Copyright 1996 Elsevier B.V.](image2)
FIG. 5. The free surface $\xi$ (thick line) and its first three spatial derivatives $\xi'$ (dotted-dashed line), $\xi''$ (dashed line), and $\xi'''$ (continuous line). $r_j$ is the jump position defined numerically, the jump definition $\xi'' = 0$ is marked with a point. Numerical solutions were obtained using $Q = 6.32 \times 10^{-1} \text{ cm}^3/\text{s}$, $L = 11.57 \text{ cm}$ in the water case. Variables are presented in the non-dimensional notation.

independently $T_0^{(0)} = 0$ and $T_1^{(0)} = 0$, that gives a cubic equation in $\epsilon_j$,

$$\epsilon_j^3 - a_1 \epsilon_j - a_0 = 0 \quad \text{and} \quad r_j^2 = \frac{9 \text{Re}}{35} \epsilon_j,$$

where $a_1 = \frac{27}{30} \frac{c_1 F}{\text{Re}} (r_j^2 + \frac{c_1}{\text{Re}})^{-1}$ and $a_0 = \frac{27}{30} F \left( r_j^2 + \frac{c_1}{\text{Re}} \right)^{-1}$. In the function $a_1$ the factor $c_1 = \delta \xi / \delta r$ has been neglected, but it may be used as a free parameter. The first approximation for $r_j$ is obtained by replacing $\epsilon_j = 1 - \delta \epsilon_j$ and taking the leading order in $\delta \epsilon_j$ that gives $r_{0,0} = (9 \text{Re}/35)^{1/2}$. Using the Reynolds number definition, we return to the dimensional variables and we obtain an scaling law that is compared with experimental data in Fig. 6,

$$r_{0,0} = \left( \frac{9 Q h}{35 \pi v} \right)^{1/2}.$$

FIG. 6. SCalings for the circular hydraulic jump. Experimental data$^{15}$ (points) for 3 different viscosities: $\nu_w$ (water viscosity, circles), $v_1 = 15 \nu_w$ (squares), $v_2 = 95 \nu_w$ (rhomboids) compared with $r_{0,0}$ (dotted-dashed line, Eq. (12)), $r_1$ (dashed line, solution of Eq. (15)), $r_0$ (thick dashed line, Eq. (16) (right), and $r_{0,1}$ (thick dotted-dashed line, Eq. (17)). Hydraulic jump radii are scaled with $L = 16.5 \text{ cm}$ (plate radius), the fluxes are scaled with $Q = 100 \text{ cm}^3/\text{s}$. Reproduced by permission from Hansen et al., Phys. Rev. E 55, 7048–7061 (1997). Copyright 1997 American Physical Society.
1. First approach

One of the reasons that may explain the mismatch between (12) and the experimental results\(^{15}\) is, supposing that \(h_0 = h\) at the jump position. A good approximation for the jump height \(h_0\) in Eq. (11) (right) could improve the results. Here, we use the critical height\(^{22}\) \(h_c = (Q_m^2/\gamma g)^{1/3}\). \(Q_m\) is defined as the mass flux and it relates subcritical and supercritical velocities and heights as \(Q_m = v_1 h_1 = v_2 h_2\). We define \(h_1\) and \(v_1\) as the supercritical height and velocity, respectively. \(h_2 = h\) and \(v_2\) are the fluid height and velocity downstream the jump. The critical height \(h_c\) satisfies the following inequality \(h_1 < h_c < h_2\). As the given flux \(Q\) is related to the supercritical velocity, we have \(v_1 = Q/A_1\) where \(A_1 = 2\pi r_1 h_1\), replacing this into \(Q_m\), the critical height is

\[
h_c = \left(\frac{Q^2}{4\pi^2 r_1^4 \gamma g}\right)^{1/3}.
\] (13)

Equation (13) is replaced into (11) (right) in the place of \(h_0\) and \(r_0\) is replaced by \(r_1\). After some algebra, we recover the scaling law (1) obtained in Ref. 9 with a determined coefficient \(2^{-1/4}(9/35)^{3/8}\gamma^{-5/8}\). This approximation is better, nevertheless it does not depend on the surface height \(h\). Final fluid depth dependency on the jump radii has been indicated in experimental reports as Refs. 11 and 12. Now we will use the supercritical height \(h_1\) (see Fig. 1) in (11) (right), that could be a better approximation of the fluid depth at the actual jump position, defined as \(r_j\) by the farthest point from the jet where \(\xi'(r) = 0\) holds (see Fig. 5). The supercritical depth \(h_1\) is closer to the exact jump height \(\xi_j\), which is obtained from numerical solutions. \(h_1\) is found using a correction of the Bélanger equation\(^{23,24}\) for non-viscous fluids with surface tension terms,\(^{25}\) derived in cylindrical coordinates\(^{13,26}\)

\[
h_1 = \frac{h}{2} \left(1 + \frac{2}{B_2}\right) \left[\sqrt{1 + \frac{8F^2}{(1 + 2/B_2)^2}} - 1\right] \approx \frac{2hF^2}{1 + 2/B_2} + O(F^4),
\] (14)

where \(F\) is the Froude number in the subcritical region and \(B_2 = \rho gh r_j/\gamma\). When the surface tension is neglected, \(B_2 \to \infty\) and Eq. (14) reduces to the familiar jump condition.\(^{23,24}\) In Ref. 26, we replace \(v_j\) by \(r_1\) into \(F\), which is defined according to that reference (Eq. (2.5)) as \(F = (3Q^2/10\pi^2 \gamma r_1^3 h^3)^{1/2}\), and using that the Froude number may be written as \(F = (h_c/h)^{3/2}\), the critical height is \(h_c = (3Q^2/10\pi^2 \gamma r_1^3 h)^{1/3}\), which is slightly different from (13). We will keep using definition (13). As \(h_c < h\), we expand (14) for low Froude numbers. At second order in \(F\), and replacing \(h_0\) by \(h_1\) and \(r_0\) by \(r_1\) in (11) (right), we obtain a fourth-order polynomial equation

\[
r_1^4 + \beta r_1^3 = \alpha,
\] (15)

where \(\alpha = 9Q^3/70\pi^3 \gamma h^2 g\) and \(\beta = 2\gamma/\rho gh\). The solution of (15) is shown in Fig. 6 compared with data obtained from Ref. 15. The surface tension correction is noticed in particular at low flux in water (see Fig. 6). In (15), we have used an equation for non-viscous fluids (14) combined with (11) (right) then the result do not match the data at highly viscous fluids. In the no surface tension limit (\(\beta \to 0\)), we obtain \(r_1 \approx \alpha^{1/4}\) or

\[
r_1 \approx \left(\frac{9}{70 \pi^3 \gamma h^2 g}\right)^{1/4}.
\]

The complete solution of (15) is plotted in Fig. 7, matching the first experimental points (circles). The last points (squares) are the jump radii of curves type II, where a second roll is observed (see Fig. 3). These effects are related to the transition to the polygonal regime where angular and high order terms should be taken into account.

2. Second approach

Section III B 1 shows how to obtain scaling laws from the circular hydraulic jump equations (3), the critical height \(h_c\) and the jump condition (14). Although these laws are in agreement with experimental results, they are not completely obtained from our approach, and (15) is not accurate
for highly viscous fluids. Solving the leading order of (11) in $\epsilon$, $\epsilon_0 \to \epsilon_0$, $r_0 \to r_0$ we have

$$\epsilon_0 = \left(\frac{3F}{2Re}\right)^{1/4} \quad \text{and} \quad r_0 = \left(\frac{9Re}{35} \frac{27F}{70}\right)^{1/8}. \quad (16)$$

The latter equation for $r_0$ in dimensional variables is the scaling law (1) found in Ref. 9 with a precise coefficient derived from Eqs. (3), $3(3/2\pi^{5/2}5^4)^{1/3}$. Equation (16) is plotted in Fig. 6 compared with experimental results. This scaling law matches experimental data, nevertheless, it does not depend on physical parameters as fluid height or surface tension. We will now solve (11). As coefficients in (11) depend on $r_0(\epsilon_0)$, we replace $r_0$ by $r_{0,0}$ in $a_1$ and $a_0$ and solve for $\epsilon_0$, which is called $\epsilon_{0,1}$. Equation (11) has 3 solutions and we take the solution that gives result (16) in the limit $\epsilon \to 0$. Replacing the correct solution of $\epsilon_{0,1}$ into (11) (right) we obtain

$$r_{0,1} = \sqrt{\frac{18Re}{35} \frac{3^{1/3}a_{1,0} + 2^{-2/3}(9a_{0,0} + \sqrt{-12a_{1,0}^3 + 81a_{0,0}^2})^{2/3}}{6^{2/3}(9a_{0,0} + \sqrt{-12a_{1,0}^3 + 81a_{0,0}^2})^{1/3}}} \quad (17),$$

where $a_{1,0} = \frac{27\epsilon a}{20}\left(r_{0,0}^2 + \frac{v}{Re}\right)^{-1}$ and $a_{0,0} = \frac{27\epsilon}{70}\left(r_{0,0}^2 + \frac{v}{Re}\right)^{-1}$. This is compared with experimental data of the jump radius as a function of the final fluid depth with $r_0$ and $r_{0,1}$ in Fig. 7. $r_{0,0}$ is not shown because it grows as a function of $h^{1/2}$. The circles in Fig. 7 are the jump radii of type I profiles, where a separation eddy is located close to the shock position, at the subcritical region. The squares are the type II profiles where a second surfing roll is observed (see Fig. 3). These effects are related to the transition to the polygonal states, where angular and high order terms are important. In this work, we just consider the circular regime and type I jumps.

As a conclusion we compare the obtained results. The first approach is derived using Ref. 14 and Eq. (14). The latter equation is obtained from non-viscous fluid equations, then the outcome does not depend strongly on the viscosity. It provides very good agreements with experimental data but is not accurate at highly viscous fluids (95 times more viscous than water in this case). The second approach is completely derived from Ref. 14 and provides good agreements with experimental measures as a function of the flux, but is less accurate when comparing with the final depth. This last problem will be investigated in order to improve the outcome, which is related to higher order equations (see, for instance, Eq. (11)). The results concern very small Reynolds numbers, for viscous and thin fluid layers as mentioned previously.

FIG. 7. Radii dependence with outer fluid depth for $Q = 0.031$ l/s, $v = 0.97 \times 10^{-3}$ m$^2$/s. Experimental results (type I curves (circles) and type II curves (squares)), $r_1$ (dashed line, solution of Eq. (15)), $r_0$ (thick dashed line, Eq. (16) (right)), and $r_{0,1}$ (thick dotted-dashed line, Eq. (17)) compared. Jump radii are normalized with $L = 37.9$ mm and depths with $h = 2.77$ mm. Reproduced by permission from Bohr et al., Physica B 228, 1–10 (1996). Copyright 1996 Elsevier B.V.
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21. As the final height is not provided for the surface velocity in Refs. 11 and 12, we assume a final depth higher than 2.77 mm as the height difference \(d_0\) from the plate to the top of the rim is \(d_0 = 1.1\) mm.
25. In this work we use \(\zeta = 1\), in the limit of abrupt jumps (strong slope jumps).