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We propose a model that captures the dynamics of a carnivorous plant, *Utricularia inflata*. This plant possesses tiny traps for capturing small aquatic animals. Glands pump water out of the trap, yielding a negative pressure difference between the plant and its surroundings. The trap door is set into a meta-stable state and opens quickly as an extra pressure is generated by the displacement of a potential prey. As the door opens, the pressure difference sucks the animal into the trap. We write an ODE model that captures all the physics at play. We show that the dynamics of the plant is quite similar to neuronal dynamics and we analyse the effect of a white noise on the dynamics of the trap.

**Keywords:** bladderwort; carnivorous plant; fast motion; excitable dynamics; *Utricularia*

1. INTRODUCTION

*Utricularia inflata* is an amazing, aquatic, rootless plant. At first sight, they are long stems topped by yellow flowers, but their underwater leaves are endowed with millimetre-sized traps. These traps are ingenious systems developed by the plant in order to survive in nutrient-poor habitats. When a prey comes too close to the trap, it touches some trigger hairs (figure 1a). This mechanical stimulus allows the opening of the trap: the door buckles, slides inwards and finally unlocks. Thus, the prey and its surrounding water are quickly sucked into the inflating trap (figure 1c). Then, *Utricularia* deflates slowly, thanks to the activity of membranar bifid glands, which actively pump the water out of the trap. Actually, the pumping is based on the transport of chloride ions, which creates a local osmotic gradient in the trap membrane and an accompanying flux of water [1,2]. As a result, the trap deflates and returns into a capturing configuration (figure 1b).

In order to fully understand the mechanical behaviour of this trap, we built a full dynamical model. Indeed, previous studies focus only on the mechanics of the trap body [3], or on the door [4] but no model was proposed to link those two aspects. Here, we introduce the hydrodynamics in order to couple these two elastic parts. We present our model in §2 by linking the internal pressure and the position of the door with a set of two coupled differential equations. Our dynamical model predicts a wide range of behaviours, including excitability leading to a fast suction, and spontaneous or periodic firings, which are all observed experimentally on *Utricularia* [5]. Note that our purely mechanical system is described by a set of equations that are similar to that of other models developed for electrically excitable media, such as the FitzHugh–Nagumo model for spike generation in nerves [6]. We find the relevant parameters that completely characterize the whole dynamics. Numerical results are analysed in §3. In §4, we discuss the suitability of our model by comparing our results with the real behaviour of *Utricularia*. Fluctuations are introduced in the last section, in order to account for the statistics of spontaneous firings of the trap [5].

2. DERIVATION OF THE MODEL

The mechanism for capturing prey is based on the suction of the fluid near the door, induced by the trap deformation. The bifid glands, by expelling water, build up a high pressure difference between the interior and the exterior of the trap, with a characteristic time of a hundred minutes. For a critical volume, the closed door can no longer sustain this pressure difference, and the trap inflates in a few milliseconds while sucking in the exterior fluid (with the potential prey). The existence of a critical threshold yielding an explosive response followed by a slow recovery are the signature of an excitable system, like for the neuronal dynamics [6].

We first describe the geometric properties of the trap, then we model the temporal variations of the volume and we approximate the door opening/closing dynamics. Finally, we close our model by coupling this dynamics with the volume equation.
2.1. Geometry of the trap body

The trap, which is a biconcave disc, as pictured in figures 1 and 2, can be approximated as a deformable cylinder [4] of diameter $L \approx 1.5$ mm and variable height $e$. The volume is approximated by

$$V = \pi \left(\frac{L}{2}\right)^2 e. \quad (2.1)$$

When the trap is fully inflated, $e = 0.8$ mm, and the maximum volume is $V_{\text{max}} = 1.41$ mm$^3$. As the trap is fully deflated, $e = 0.4$ mm, and the minimal volume is $V_{\text{min}} = 0.67$ mm$^3$ [4]. The area of the membrane of the trap $S_m$ is approximately given by

$$S_m = \frac{\pi}{2} L^2 + \pi L e. \quad (2.2)$$

2.2. Temporal variation of the volume

We now consider the temporal variations of $V$, as observed in [5]. Experiments show that $V$ decays exponentially to a final volume $V_0$ [4]. The fluid flux has three contributions. First, the bifid glands of the membrane actively pump the water out of the trap to decrease $V$. The flow rate corresponding to this mechanism is $-q$ with $q$ positive. The second effect is related to the porosity of the trap. The membrane pores induce a Darcy flow that is proportional to the pressure difference $\Delta P$, the surface of the membrane $S_m$ and the porosity $\delta_e$. Osmotic pressure could also be at the origin of this leakage. Finally, if the door is opened, there is an additional flux that tends to equilibrate the pressure difference between the exterior and the interior of the trap. For now, we simply write the balance of fluxes related to the membrane:

$$\frac{\partial V}{\partial t} = -q + \delta_e S_m \Delta P, \quad (2.3)$$

where $q = 2.3 \times 10^{-13}$ m$^3$ s$^{-1}$ [4] (equivalent to a transport velocity of $4 \times 10^{-3}$ m s$^{-1}$) through the surface $S_m$ and we choose $\delta_e = 2.4 \times 10^{-12}$ m$^{-1}$ Pa$^{-1}$ of the same order of magnitude as those reported for vegetal cells [7,8]. The membrane elasticity connects the volume loss to the pressure load: the pressure difference $\Delta P$ varies almost linearly with $V$ in experiments and in simulations [4]:

$$\Delta P = -d(V - V_{\text{max}}), \quad (2.4)$$

with $V_{\text{max}}$ the volume for which the bending energy of the trap is minimum. Here, $d$ mixes the effects of the elasticity and the geometry of the trap. Experimental results show that a variation from $V = V_{\text{max}}$ to $V = V_{\text{min}}$ yields a pressure difference equal to 0.15 bar [4]; so we evaluate $d = 2.027 \times 10^{-12}$ Pa m$^{-3}$. As a consequence, we conclude from (2.3) and (2.4):

$$\dot{V} = \frac{1}{\tau}(V_0 - V), \quad (2.5)$$

where $1/\tau = \delta_e d S_m$ and $V_0 = V_{\text{max}} - q\tau$ is the volume at final equilibrium.

At this stage, we can model the action of the door with the following argument: as $V$ becomes smaller than $V_c$, the door opens and $V$ is settled instantly to a given value $V_{\text{max}}$. This crude approximation
will be refined as we develop the model of the door displacement (§2.3). If at $t=0$, $V=V_{\text{max}}$, then the temporal evolution of the volume is

$$V(t) = V_0 + (V_{\text{max}} - V_0) e^{-t/\tau}. \quad (2.6)$$

Consequently, depending on the relative value of $V_0$ to $V_c$, we can observe two distinct behaviours (figure 3):

- if $V_c < V_0$, the volume of the trap converges to $V_0$. Furthermore, if $V_c \sim V_0$, the trap dynamics is similar to an excitable system [9]. A small external perturbation can potentially decrease $V$ under $V_c$, leading to the opening of the door, which in turn increases $V$ up to $V_{\text{max}}$.

- if $V_c > V_0$, the volume will oscillate between the value $V_{\text{max}}$ and $V_c$. The period of the oscillation $T_V$ can be easily obtained. At $t = T_V$, the door opens as $V = V_c$. We therefore deduce from (2.6) that

$$T_V = \tau \log \frac{V_{\text{max}} - V_0}{V_c - V_0}. \quad (2.7)$$

These two distinct behaviours have been described in Vincent et al. [5].

The critical volume $V_c$ will be assessed by studying the door dynamics.

2.3. Dynamics of the door

We model the door as an elastic circular shell of radius $R = 3 \times 10^{-4} \text{ m}$ and thickness $h = 3 \times 10^{-5} \text{ m}$ [4], with half of its edge clamped in the membrane. This door is compressed against a flat and rigid substrate, leading to a bent shape as pictured in figure 4a. In this configuration, we assume that no fluid leaks through the door, which slightly deforms to sustain an increase in pressure difference $\Delta P$. When this latter exceeds a critical value, a buckling instability occurs (figure 4b), yielding to a fast opening of the door that permits a fast entrance of the fluid that equilibrates the pressure (figure 4c). Once $\Delta P$ vanishes, the door comes back quickly to its original configuration (figure 4a), driven by the minimization of its elastic energy.

We note $Z$ the position of the centre of mass of the door. The door is subject to elastic forces, to the external load $-\pi R^2 \Delta P$ exerted by the pressure and to damping. The effective mass $m$ of the door can be approximated by

$$m = \rho \pi R^2 h + \kappa p R^3, \quad (2.8)$$

where $\rho$ measures its density (approx. equal to $\rho_{\text{water}} = 10^3 \text{ kg m}^{-3}$). The first term is the mass of the door, and the second term represents the added mass. The factor
\( \kappa \) is of order 1 [10]. The cubic dependance models the fluid volume that the door needs to displace during its motion. We consequently write:

\[
m\dot{Z} = f_{\text{elastic}} + f_{\text{damping}} - \pi R^2 \Delta P. \tag{2.9}
\]

Very roughly, the damping terms can be written with a sum of two contributions. As the door moves fast, the fluid is at a high Reynolds number and generates a drag proportional to \( \rho \dot{Z}^2 \) [11]. For small movements, the latter damping is small compared with the viscous friction, proportional to the Stokes number \( \eta R \dot{Z} \), where \( \eta \) is the dynamical viscosity of water. Consequently, we write:

\[
f_{\text{damping}} = -a \rho R^2 (\dot{Z})^2 \text{sign}(\dot{Z}) - b \eta R \dot{Z}, \tag{2.10}
\]

where the two constants \( a \) and \( b \) are of order 1, and depends on the geometry of the door. It remains to define the elastic forces. We approximate the door with a cylindrical shell, against a rigid substrate using a bending energy \( \mathcal{E} \):

\[
f_{\text{elastic}} = -\frac{\partial \mathcal{E}}{\partial \dot{Z}}, \tag{2.11}
\]

For such systems, the plate has two stable buckled equilibria separated by an unstable state. The bending energy should display two minima separated by one maximum, this is why we propose

\[
\mathcal{E} = -P_h R \frac{Z^2}{2} \left( 1 - \frac{1}{2} \left( \frac{Z}{Z_0} \right)^2 \right), \tag{2.12}
\]

where \( P_h \) is the critical pressure provoking the door buckling. In the case of a spherical shell of radius \( R \) and thickness \( h \), it writes \( P_h = Eh^2/R^2 \) [12].

The energy (2.12) predicts an unstable state \( Z = 0 \) with two stable states \( Z = \pm Z_0 \). Following figure 4, we choose the door to be closed when \( Z > 0 \) and opened when \( Z < 0 \). The door is not articulated on freely rotating hinges, as would be a real door. Actually, it is clamped to the main body by a thicker and less deformable part (figure 4). The deformation of this thicker part means that the opened state is energetically unfavoured. To account for the deformation cost of the thick part, we therefore break the symmetry \( Z \rightarrow -Z \) by adding a term proportional to \( -Z \) in the energy and we introduce a prefactor \( c \):

\[
\mathcal{E} = -P_h R \frac{Z^2}{2} \left( 1 - \frac{1}{2} \left( \frac{Z}{Z_0} \right)^2 \right) + c Z_0 Z. \tag{2.13}
\]

In figure 5, we plot the elastic energy \( \mathcal{E} \) as a function of \( Z \). The effect of the constant \( c > 0 \) renders the closed door configuration to be more stable from an energetic point of view. In particular, when \( c > 2/3\sqrt{3} \), this potential exhibits a single minimum, with \( Z > 0 \), which corresponds to the closed door state.

2.4. Mass flux induced by the door

It remains to couple the volume equation (2.6) to the door dynamics (2.9). We model the flow rate \( Q \) in the channel induced by the opened door, in the presence of a pressure difference:

\[
Q = \pi R^2 s \left( \frac{Z}{Z_0} \right) U. \tag{2.14}
\]

Here, \( \pi R^2 s (Z/Z_0) \) is the surface of the channel. The non-dimensional surface \( s (Z/Z_0) \) is approximated with

\[
s(x) = -fxH(-x), \tag{2.15}
\]

where \( H(x) \) is the unit step function. The constant \( f \) is a geometric factor.

The fluid velocity \( U \) can be assessed using Bernoulli relation [11], because the Reynolds number of the incoming flow is quite high (around 1000 [3]). We therefore write

\[
U = \text{sign}(\Delta P) \sqrt{\frac{2}{\rho} |\Delta P| s \left( \frac{Z}{Z_0} \right)}. \tag{2.16}
\]

As consequence, we deduce the flux \( Q \)

\[
Q = \text{sign}(\Delta P) \pi R^2 \sqrt{\frac{2}{\rho} |\Delta P| s \left( \frac{Z}{Z_0} \right)}. \tag{2.17}
\]

2.5. Model closure

In order to close the model, we need to couple the volume, the door position and the pressure difference. Adding the contribution of the open door (2.17) to the flux balance (2.3), we obtain

\[
\frac{\partial V}{\partial t} = -q + \delta_s m \Delta P + \text{sign}(\Delta P) \pi R^2 \sqrt{\frac{2}{\rho} |\Delta P| s \left( \frac{Z}{Z_0} \right)}. \tag{2.18}
\]

Injecting the link (2.4) between \( \Delta P \) and \( V \) into the earlier-mentioned relation, we deduce our dynamical model for the trap:

\[
\Delta P = -d \left[ -q + \delta_s m \Delta P + \text{sign}(\Delta P) \pi R^2 \sqrt{\frac{2}{\rho} |\Delta P| s \left( \frac{Z}{Z_0} \right)} \right]. \tag{2.19}
\]
and
\[
m\dddot{Z} = P_b R \left( Z - \left( \frac{Z}{Z_0} \right)^2 Z + c Z_0 \right) - \pi R^2 \Delta P - \alpha p R^2 \dot{Z}^2 \text{sign}(\dot{Z}) - \beta \dot{Z}.
\] (2.20)

We now set our system in a non-dimensional form with the scalings \( \overline{Z} = \frac{z}{Z_0}, \overline{\Delta P} = \frac{\Delta P}{P_b}, \) and \( t = \Theta_t \). Because the characteristic length scale for \( Z_0 \) is proportional to \( R \), we set \( Z_0 = R \). Using (2.20), the balance between the acceleration term with the elastic response gives the characteristic timescale \( \Theta \) of the opening door:
\[
\Theta = \sqrt{\frac{m}{P_b R}} \sim 100 \mu s,
\] (2.21)
where we used \( E = 2.7 \text{ MPa} \). \( \Theta \) is the shortest time of the system; so the non-dimensional time \( s \) is very long. As consequence, our non-dimensional system can be written as
\[
\dddot{\overline{Z}} + \alpha \ddot{\overline{Z}}^2 \text{sign}(\dot{\overline{Z}}) + \beta \ddot{\overline{Z}} = -z^3 + c - \lambda p
\] (2.22)
and
\[
\dot{p} = \frac{1}{\tau_p} (p_0 - p) - \text{sign}(p) \gamma \sqrt{|p|} s(z),
\] (2.23)
with
\[
\alpha = \frac{\alpha p Z_0 R^2}{m} \sim 0.34,
\] (2.24)
\[
\beta = \frac{\beta \eta R \Theta}{m} \sim 3.7 \times 10^{-4},
\] (2.25)
\[
\gamma = \Theta_t \left( \frac{2}{\rho P_b} \right) \sim 0.16,
\] (2.26)
\[
\lambda = \frac{\pi R}{Z_0} = \pi,
\] (2.27)
\[
p_0 = \frac{q}{\delta S_m P_b} \sim 0.56
\] (2.28)
and
\[
\tau_p = \frac{1}{\Theta_t \delta S_m} \sim 3.2 \times 10^7.
\] (2.29)

The dynamical viscosity is \( \eta = 10^{-3} \text{ Pa} \cdot \text{s} \).

3. ANALYSIS OF THE MODEL

The system (2.22) and (2.23) presents two very different timescales. The door motion is generally very fast compared with the pressure recovery time; it is reminiscent of excitable system, for which a qualitative analysis is performed using nullclines. Owing to the slow temporal variation of \( p \), the equation (2.22) is a nonlinear, damped oscillator for the variable \( z \). When \( \dot{z} = 0 \), the door accelerates positively if \( \pi p < f(z) \), with \( f(z) = -z^3 + c \), and negatively otherwise.

If \( z > 0 \), the equation (2.23) is
\[
\dot{p} = \frac{1}{\tau_p} (p_0 - p),
\] (3.1)
and the pressure increases as \( p_0 > p \), and decreases on the other case.

If \( z < 0 \), the door is opened and the fluid flow (through the opening) controls the pressure variation; the equation (2.23) becomes
\[
\dot{p} = -\text{sign}(p) \gamma \sqrt{|p|} s(z).
\] (3.2)

In consequence, if \( z < 0 \) and \( p > 0 \), the pressure difference \( p \) decreases whereas if \( z < 0 \) and \( p < 0 \), \( p \) increases. These behaviours can be summarized in figure 6, where we plot the functions \( p(z) \) at which \( \dot{p} = 0 \) and \( \dot{z} = 0 \), defining the two nullclines.

The temporal evolution of \( p \) is very small compared with \( z \), except in the close neighbourhood of \( p = (z - z^3 + c) / \pi \).

3.1. Stationary states

In this section, we investigate the stationary states \((z_s, p_s)\) of the model, and we need to solve the following system of equations:
\[
\frac{1}{\tau_p} (p_0 - p_s) - \text{sign}(p_s) \gamma \sqrt{|p_s|} s(z_s) = 0
\] (3.3)
and
\[
\frac{z_s - z_s^3 + c}{\pi} = p_s.
\] (3.4)

They can be guessed by studying the intersections of the two curves plotted in figure 6. Because the temporal evolution of \( p \) is very small compared with those of \( z \), we can assess the linear stability of the fixed points by using the equation (2.22) with \( p = p_s \); then the evolution of the perturbation \( \tilde{z} \) around \( z_s \) obeys to
\[
\ddot{\tilde{z}} + \beta \dot{\tilde{z}} = f'(z_s) \tilde{z}.
\] (3.5)

As a consequence, the stationary points \( z_s \) will be stable if \( f'(z_s) < 0 \), because \( \beta \) is always positive.

Depending on the signs of \( z_s \) and \( p_s \), we meet two cases:

- \( z_s > 0 \), the fixed points are \( p_s = p_0 \) (from (3.1)) and the roots \( z_s \) of \( \pi p_0 = z_s - z_s^3 + c \). This last equation has two real solutions if \( c < \pi p_0 < c + 2/3 \sqrt{3} \).

A close look at figure 6 permits us to conclude that only the highest root is stable. If \( p_0 < c / \pi \),
then the system has only one stable stationary point. Perturbations of the door position around these stable states will create damped oscillations, while \( p \) will decrease very slowly to \( p_0 \).

- \( z < 0 \): in such a case, the flow through the door dominates the dynamics for the pressure. Therefore, \( p_0 = 0 \) (from equation (3.2)). It is necessary to solve \( z = z_0 = z^2 + c = 0 \), which admits two real negative solutions only if \( c < 2/3\sqrt{3} \). Again, figure 6 predicts that only the most negative root will be stable.

### 3.2. Phase diagram

The previous study of stationary states and their linear stability yields the construction of the phase diagram of the trap. Depending on the number of stable states, we distinguish four regions (figure 7):

- Region A and A': the only stable state is the closed door. Notice that when the pumping generates a pressure difference close to spontaneous buckling, i.e. when \( \pi p_0 \leq c + 2/3\sqrt{3} \), the system will exhibit a strong sensitivity to a variation of the pressure or to the position of the door. It is in this regime that the system is understood as excitable. We made a simulation of the dynamics in this case, as shown in figure 8: a small increment of pressure will lead to a quick opening of the door, producing damped oscillations while it comes back to the closed state, and that the bifurcations glands pump the fluid out. The pressure difference equilibrates to \( p_0 \), and the trap will be ready to capture another prey.

- Region B and B': in this range of parameters, the trap has two stable steady states: depending on initial conditions, the door remains opened or closed. An example of these dynamics is shown in figure 9.

- Region C: the only stable state is the door opened. An example of these dynamics is shown in figure 10.

- Region D: because the trap does not offer stable stationary points, it undergoes relaxation oscillations between \( \pi \omega_{\text{min}} = f(-1/\sqrt{3}) = c - 2/3\sqrt{3} \) and \( \pi \omega_{\text{max}} = f(1/\sqrt{3}) = c + 2/3\sqrt{3} \). An example of these dynamics is shown in figure 11.

### 4. DISCUSSION

#### 4.1. Comparison with experiments

Experimentally, the trap presents the three states: excitable, metronomic and dead, described in §3. If the plant is dead, the trap is not functioning anymore, and the trap remains opened as shown in the region C of figure 7. In normal conditions, the trap is in state A of figure 7, with \( \pi p_0 \leq c + 2/3\sqrt{3} \): a small amount of extra pressure can force the door to open, yielding the suction of some liquid around it. For the simulations presented in figure 12, the variation in pressure difference rises to approximately 16 kPa, with the parameter values proposed in §2.5. This perfectly agrees with the experimental value of 10–20 kPa [4]. The duration for the trap to come back to its stable state is evaluated by \( \tau \approx 54 \text{ min} \), which fairly matches the experimental values of 25–50 min [5]. Finally, we numerically find (figure 10) that the time for the door buckling is around 3.6 ms, namely in the same order of magnitude as those reported in Vincent et al. [5].

Obviously, our simple model does not pretend to be accurate enough to exactly match the experiments; however, our non-dimensional parameters can be fit to experimental measures. The model shows that the trapping function is obtained only in one regime, when the mechanical parameters of the trap are in a specific range.

#### 4.2. Effect of noise on the system

This highly sensitive trigger renders the mechanism for capturing preys very efficient; nevertheless, we shall investigate the effect of noise. In fact, for high enough amplitudes, some noise induces stochastic trapping events because of the excitable behaviour.

The relaxation oscillations observed in the D region of the phase diagram have been recently described as spontaneous firings [5]. To investigate the effect of the noise, we consider the membrane as \( N \) interconnected ‘particles’ submitted to an external noise. Accordingly, the positions of the particles follow the Langevin equations:

\[
\forall 1 \leq j \leq N, \partial_t r_j = \frac{D_j}{b_j} f_j + \eta_{b_j},
\]

where \( D_j \) is the diffusion coefficient, \( b_j \) is the Boltzmann constant, \( T \) is the temperature, \( f_j \) is the external force on the \( j \)th particle. The Gaussian random drifts are defined by

\[
\langle \eta_{b_j}(t) \rangle = 0
\]

and

\[
\langle \eta_{b_j}(t) \eta_{b_j}(\ell) \rangle = 2D_j \delta(t-\ell).
\]

This description allows us to access the stochastic fluctuations of the volume \( V \) (see appendix A for details):

\[
\dot{V} = \frac{1}{\tau} (V_0 - V) + A_m(t),
\]

where \( 1/\tau = \delta_v dS_m \) and \( V_0 = V_{\text{max}} - \gamma t \) is the trap volume at equilibrium. The stochastic forcing for the
volume is defined by

\[
\langle A_m(t) \rangle = 0
\]

and

\[
\langle A_m(t) A_m(t') \rangle = 2 \phi D_L \delta(t - t'),
\]

where \( \phi D_L \) is a macroscopic volume diffusion coefficient computed in appendix A.

We need to compute the minimum volume \( V_c \) that the trap can sustain, corresponding to the highest pressure difference \( \Delta P_c \) that the system can maintain:

\[
\Delta P_c = \frac{P_b}{\pi} \left( c + \frac{2}{3 \sqrt{3}} \right).
\]

The critical volume is straightforwardly deduced from the relation (2.4):

\[
V_c = V_{\text{max}} - \frac{\Delta P_c}{d}.
\]

If \( V < V_c \), the doors open and the trap volume instantaneously reaches \( V_{\text{max}} \).

We translate (4.3) with the following change of variable:

\[
x = \frac{V - V_c}{V_{\text{max}} - V_c},
\]

\[
z_0 = \frac{V_0 - V_c}{V_{\text{max}} - V_c}.
\]
Then, (4.3) becomes:

$$x(t) = x_0 + e^{-t/\tau} \left( x_{\text{max}} - x_0 + \int_0^t A(s) e^{s/\tau} ds \right).$$

(4.12)

where the initial condition is $x(0) = x_{\text{max}}$, i.e. $V(0) = V_{\text{max}}$. The Fokker–Planck equation [14] describes the time evolution of the probability density function of $x$; it can be written as follows:

$$\partial_t c(x, t) = \frac{1}{\tau} \partial_x ((x - x_0)c) + D \partial_x^2 c,$$

(4.13)

where

$$D = \frac{\phi D_L}{(V_{\text{max}} - V_c)^2}.$$  

(4.14)

The dimension of $D$ is $s^{-1}$. Here we take the diffusion coefficient for a bilipid: $D_L = 10^{-11} \text{m}^2 \text{s}^{-1}$ [15]. The initial condition at $t = 0$ is $c(x, 0) = \delta(x - x_{\text{max}})$, and the boundary conditions are $c(0, t) = c(\infty, t) = 0$. We focus on the system behaviour for $x = 0$. In fact, if $x = 0$ ($V = V_c$), the value of the volume is instantaneously reset to $V_{\text{max}}$. The probability flux $j(t)$ at $x = 0$ is

$$j(t) = D \partial_x c(x, t) |_{x=0}.$$  

(4.15)

By integrating this flux over the whole temporal domain, we deduce the probability $\epsilon$ that the system escapes at $x = 0$:

$$\epsilon = \int_0^\infty j(t) dt.$$  

(4.16)

We note $\hat{c}(x, s)$ the Laplace transform of $c(x, t)$. By performing the Laplace transform of (4.13) and by imposing the initial boundary condition, we obtain

$$-\delta(x - x_{\text{max}}) + sc(x, s) = \frac{1}{\tau} \partial_x ((x - x_0)\hat{c}(x, s)) + D \partial_x^2 \hat{c}(x, s).$$

(4.17)

The solution $\hat{c}$ of the earlier-mentioned equation allows us to compute the Laplace transform $\hat{j}(s)$ of the probability flux:

$$\hat{j}(s) = D \partial_x \hat{c}|_{x=0}.$$  

(4.18)

Because $\hat{j}(s) = \int_0^\infty j(t) e^{-st} dt$, we deduce the following
We can evaluate the average first exit time \( \langle t \rangle \) using (4.20):

\[
\langle t \rangle = -\tau \left( \partial_y H_z \left( \frac{x_{\text{max}} - x_0}{\sqrt{2D\tau}} \right) - \partial_y H_z \left( -\frac{x_0}{\sqrt{2D\tau}} \right) \right)_{X = 0},
\]

(4.21)

where \( H_z(y) \) is the Hermite polynomial of degree \( z \). As shown in figure 13, for low noise, the stochastic period of the oscillations given by (4.21) tends to the deterministic period \( T_V \) (2.7). In terms of the variable \( x \):

\[
T_V = \tau \log \left( -\frac{x_{\text{max}} - x_0}{x_0} \right).
\]

(4.22)

As \( x_0 \) becomes negative, the system exhibits oscillations. In the simulation of the metronomic system (region D) displayed in figure 11, \( x_0 = -0.2023 \) and we find \( T_V \sim 95 \text{ min} \), which is in good agreement with [5] (from 45 min to several hours). In figure 13, we see that the fluctuations reduce the oscillation period.

In the excitable regime, \( x_0 \) is small. In figure 14, we can see that the noise extends the periodic behaviour for \( x_0 > 0 \). Unsurprisingly, an increasing noise intensity shortens the period.

5. CONCLUSION

The model is found to capture all the features of the mechanical system of \textit{Utricularia} traps.

On the basis of simple mechanical ingredients, non-linear elasticity and fluid coupling, the present analysis could be useful for other mechanical systems exhibiting oscillations, in the plant or animal kingdoms.

The precise quantification of all noise sources (physiological variations, fatigue, mechanism of pumping, variability in door elasticity or closure, etc.), is yet to be analysed, and opens perspective for future research.

APPENDIX A. STOCHASTIC VOLUME EQUATION

A.1. Statistical description of the membrane

Let us assume that the membrane may be described by a collection of \( N \) interconnected ‘particles’. The positions of those ‘particles’ follow the Langevin equations that we write in the barycentric coordinates (for the sake of simplicity) as

\[
\forall 1 \leq j \leq N, \partial_t r_j = \frac{D_L}{k_b T} f_j + \eta_{b,j},
\]

(A 1)

where \( f_j \) is the external force on the \( j \)th particle and where the Gaussian random drifts are defined by

\[
\begin{align*}
\langle \eta_{b,j}(t) \rangle &= 0 \\
\langle \eta_{b,j}(t) \eta_{b,j}(t') \rangle &= 2D_b \delta(t - t').
\end{align*}
\]

(A 2)

The gyruisdom \( R_g \) of the membrane is defined by

\[
R_g^2 = \frac{1}{N} \sum_j r_j^2.
\]

(A 3)

This is a measure of the statistical expansion of the membrane. That gyruisdom allows us to define the volume of gyration of the membrane as

\[
V_g = \frac{4\pi}{3} (R_g^2)^{3/2} = \frac{4\pi}{3N^{3/2}} \left( \sum_j r_j^2 \right)^{3/2},
\]

(A 4)

which is the average statistical volume enclosed by the
membrane. Accordingly,
\[ \partial_t V_g = \frac{4\pi}{N} \left( \sum_j r_j^2 \right)^{1/2} \left( \sum_j r_j \partial_j \right), \]  
(A5)

then
\[ \partial_t V_g = \frac{4\pi}{N} R_g \left( \sum_j r_j \eta_{b_j} + \sum_j \frac{D_t}{k_0 T} f_j r_j \right). \]  
(A6)

### A.2. Fluctuations near the resting volume

Normally, we expect the Gaussian noise to be negligible against the external forces. So we are interested only in the case where the sum of the external forces is about zero on each ‘particle’, that is when the second term in (A6) tends to vanish. If we assume that we have small fluctuations, then we can approximate that all the \( r_j \) are near their initial values. Consequently, \( r_j \eta_{b_j} \approx r_j \eta_k \), where \( \eta_k \) is the projection of a Gaussian random drift on a constant direction, so that
\[ \langle \eta_k(t) \rangle = 0 \quad \text{and} \quad \langle \eta_k(t) \eta_{k_0} \rangle = 2D_t \delta(t - \tau). \]  
(A7)

By the statistical summation of Gaussian random variables, we obtain
\[ \frac{1}{N} \sum_j \langle r_j \rangle \eta_k(t) = R_g \eta_k \]  
(A8)

where \( \eta_k \) possesses the same properties than one of the \( \eta_{b_j} \). Finally, the evolution of the volume can be approximated by
\[ \partial_t V_g \approx 4\pi \left( R_g^2 \eta_k + R_g \sum_j \frac{D_t}{k_0 T} f_j r_j \right). \]  
(A9)

The first (stochastic) term can be expressed as an overall Gaussian noise. Then the linearization of the second (deterministic) term near the resting volume \( V_{g,0} \) shall mandatorily produce a first-order relaxation. This yields
\[ \partial_t V_g = \frac{1}{\tau} (V_{g,0} - V_g) + A_g(t), \]  
(A10)

where \( A_g(t) \) is a Gaussian volume drift with
\[ \langle A_g(t) \rangle = 0 \quad \text{and} \quad \langle A_g(t) A_g(t') \rangle = 2S_g^2 D_t \delta(t - t'), \]  
(A11)

where \( S_g = 4\pi R_g^2 \) is by definition the gyration surface.

### A.3. Application to Utricularia

The equation for the trap may be deduced by
\[ \partial_t V_m \approx \frac{1}{\tau} (V_{m,0} - V_m) + \frac{V_m}{V_g} A_m(t) \]  
(A12)

with
\[ \langle A_m(t) \rangle = 0 \quad \text{and} \quad \langle A_m(t) A_m(t') \rangle = 2\frac{V^2}{V_g} S_m^2 D_t \delta(t - t'), \]  
(A13)

\[ \langle A_m(t) A_m(t') \rangle = 2\frac{V^2}{V_g} S_m^2 D_t \delta(t - t'), \]  
(A14)

and
\[ \phi = \left( \frac{9V^2}{R^2} \right). \]  
(A15)

Taking into account the cylindrical geometry
\[ R_g^2 = \frac{I_g^2}{S_m}, \]  
(A16)

with
\[ I_g^2 = 2\pi \left[ 2 \int_0^{L/2} \left( r + \frac{e}{2} \right)^2 r dr + L \int_0^{L/2} \left( \frac{L}{2} + z \right)^2 dz \right]. \]  
(A17)

This crude modelling allows us to connect a microscopic particular diffusion coefficient \( D_L \) to a macroscopic volume diffusion coefficient \( 2\phi D_L \).

### APPENDIX B. COMPUTATION OF THE PROBABILITY FLUX

We first solve (14.17). The presence of the dirac function localized at \( x = x_{\text{max}} \) suggests to consider two intervals. On the first interval \([0, x_{\text{max}}]\), we compute the solution \( \tilde{c}_{a}(x, s) \), whereas we note \( \tilde{c}_{b}(x, s) \) the solution on the interval \([x_{\text{max}}, \infty]\), such that we have to solve the following equation:
\[ s \tilde{c}_{a,b}(x, s) = \frac{1}{\tau} \partial_x (x - x_0) \tilde{c}_{a,b}(x, s) + D \tilde{c}_{a,b}(x, s). \]  
(B1)

We find the asymptotic solution:
\[ \tilde{c}_{a,b}(x, s) \approx e^{- (x - x_0)^2 / 2D\tau}. \]  
(B2)

The sought solutions of (B1) are written:
\[ \tilde{c}_1 = i_1(s) f_1(x, s) + i_2(s) f_2(x, s), \]  
(B3)

\[ f_1(x, s) = e^{- (x - x_0)^2 / 2D\tau} H_{\text{norm}} \left( \frac{x - x_0}{2\tau} \right) \]  
(B4)

and
\[ f_2(x, s) = e^{- (x - x_0)^2 / 2D\tau} F_1 \left( \frac{s + 1}{2}; \frac{1}{2}; -\frac{(x - x_0)^2}{2\tau} \right) \]  
(B5)

The index \( i \) stands for \( a, b \). \( H_{\text{norm}}(z) \) is the Hermite polynomial of degree \( n \) and \( F_1 \) is the Kummer confluent hypergeometric function. Note the similarity of the function \( f_1 \) with the wave function of the quantum harmonic oscillator. The function \( f_2(x, s) \) behaves as \( 1/x \), as \( x \) tends to infinity, and this slow convergence to zero imposes that \( b_2 = 0 \), because we aim in deriving a finite density probability. As consequence we have
three unknowns ($a_{1,2}$ and $b_1$) for the three boundary conditions:

\[ \dot{c}_a(0, s) = 0, \quad (B\ 6) \]

\[ \dot{c}_a(x_{\text{max}}, s) = \dot{c}_b(x_{\text{max}}, s), \quad (B\ 7) \]

and

\[ \dot{c}_b(x_{\text{max}}, s) - \dot{c}_a(x_{\text{max}}, s) = -\frac{1}{D}, \quad (B\ 8) \]

The last condition is the consequence of the presence of the dirac function located at $x = x_{\text{max}}$. The inversion of system (B 6)–(B 8) gives

\[ a_1 = \frac{f_1(x_{\text{max}}, s)f_2(s, 0)}{Df_1(s, 0)w(x_{\text{max}}, s)}, \quad (B\ 9) \]

\[ a_2 = -\frac{f_2(x_{\text{max}}, s)}{Dw(x_{\text{max}}, s)}, \quad (B\ 10) \]

and

\[ b_1 = \frac{f_1(x_{\text{max}}, s)f_2(s, 0) - f_1(s, 0)f_2(x_{\text{max}}, s)}{Df_1(s, 0)w(x_{\text{max}}, s)}, \quad (B\ 11) \]

where $w(x, s) = f_2(x, s)\partial_x f_2 + f_1(x, s)\partial_x f_1(x, s)$ is the Wronskian of the equation (B 1). It obeys the equation

\[ DO_x w + \frac{1}{\tau} (x - x_0) w = 0, \quad (B\ 12) \]

which has as a solution $w(x, s) = Ae^{-(x-x_0)^2/2Dt}$. Consequently, we deduce that the probability flux defined in (4.18) takes the following form:

\[ \dot{j} = \frac{f_1(x_{\text{max}}, s)w(0, s)}{f_1(0, s)w(x_{\text{max}}, s)}. \quad (B\ 13) \]

Furthermore, we can compute the probability $\epsilon$ that the system escapes at $x = 0$ in 1 because

\[ \epsilon = \frac{f_1(x_{\text{max}}, 0)w(0, 0)}{f_1(0, 0)w(x_{\text{max}}, 0)}. \quad (B\ 14) \]

and $H_0(x) = 1$. This means that a noisy trap shall always fire.

REFERENCES