Nonlinear Faraday waves at low Reynolds numbers

N.O. Rojas \textsuperscript{a}, M. Argentina \textsuperscript{a,\*}, E.A. Cerda \textsuperscript{b}, E. Tirapegui \textsuperscript{c}

\textsuperscript{a} Université de Nice Sophia Antipolis, UFR Sciences, Laboratoire J. A. Dieudonné Parc Valrose - 06108 Nice Cedex 2, France
\textsuperscript{b} Departamento de Física, Universidad de Santiago, Av. Ecuador 3493, Santiago, Chile
\textsuperscript{c} Departamento de Física, Universidad de Chile, Facultad de Ciencias Físicas y Matemáticas, Av. Blanco Encalada 2008, Santiago, Chile

\textbf{A R T I C L E  I N F O}

Article history:
Received 23 December 2008
Received in revised form 16 March 2009
Accepted 18 March 2009
Available online 27 March 2009

Keywords:
Faraday waves
Nonlinear dynamics
Faraday waves
Low Reynolds Number
Free surface flows

\textbf{A B S T R A C T}

We derive a set of nonlinear amplitude equations that predicts the presence of nonlinear patterns in a vertically oscillated one dimensional viscous fluid layer.

The basic configuration is the rest state \( \mathbf{v}(x, z, t) = 0 \) with flat interface and hydrostatic pressure \( p_0 = p_0 - pg(t)(z - h) \), where \( p_0 \) is the atmospheric pressure, \( \rho \) the mass density of the fluid and \( g(t) \) is the vertical acceleration induced by the gravity and the periodic motion of plate (see Fig. 1). Since the fluid velocity is much smaller than the sound velocity we use the incompressible Navier–Stokes equations:

\begin{equation}
\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\nabla \pi + \nu \nabla^2 \mathbf{v}, \quad \nabla \cdot \mathbf{v} = 0,
\end{equation}

where \( \mathbf{v} \) is the kinematic viscosity and \( \pi = (p - p_0)/\rho \) is related to the deviation from the hydrostatic pressure. We consider a single frequency forcing \( g(t) = g[1 + \cos \Omega t] \), with \( \Omega = \text{At}^2/g \) the ratio between the forcing acceleration and the gravitational one.

In order to use the lowest number of parameters, we scale all the variables. The horizontal coordinate is scaled with \( \bar{x} = x/L \). We choose \( L \) being in the order of magnitude of the wavelength of patterns at the surface \( \lambda = \sqrt{\nu / \Omega} \), that also scales as the viscous length of the forced fluid [11,12].

The vertical coordinates are naturally scaled with the height of the fluid as \( \bar{z} = (z - \bar{h})/h \), where the tilde denotes dimensionless quantities. For the velocities, the scaling are \( \bar{v}_x = v_x/L \) and \( \bar{v}_z = v_z/h \), where \( \Omega \) is the forcing frequency. Then, the rescaled time and \( \bar{t} = \Omega t \) and \( \bar{h} = h^2 \pi / \nu \Omega L^2 \). The substitution of these scalings into the bulk equations and boundary conditions yields a set of dimensionless equations: (the tilde notation is henceforth dropped)

\begin{align}
\text{Re} \partial_t \bar{v}_x + (\bar{v} \cdot \nabla) \bar{v}_x &= - \partial_x \pi + \left( \partial_x^2 + \partial_z^2 \right) \bar{v}_x, \\
\varepsilon^2 \text{Re} \partial_t \bar{v}_z + (\bar{v} \cdot \nabla) \bar{v}_z &= - \partial_z \pi + \varepsilon^2 \left( \partial_x^2 + \partial_z^2 \right) \bar{v}_z,
\end{align}

\textbf{1. Introduction}

Faraday waves are nonlinear oscillations that appear on the surface of a fluid which is vertically and periodically accelerated [1]. This phenomenon has been extensively studied in the last decades. Experiments show plenty of structures such as squares, rhomboids, hexagons [2–4], quasipatterns [4,5,7], solitary waves [4,6–8] and transition to spatio-temporal chaos [9]. Theoretical studies have been devoted to the linear analysis in viscous fluids [10–12], amplitude equations in the weakly nonlinear regime [13–15] and phenomenological models [16,17].

In [11,12], a linearized model has been derived from the Navier–Stokes equations, but the nonlinear saturation is not addressed. In this work, we derive the nonlinear equations that govern the phenomena for thin films of viscous fluids at low Reynolds numbers. We show that the linear Mathieu equation [12] is contained in our model and for highly viscous fluids, we get the Reynolds equation [18,19].

\textbf{2. Calculation}

We consider a rigid plate covered by a thin fluid of layer \( h \) embedded in a system with horizontal and vertical coordinates \( x, z \) respectively. The velocity field is \( \mathbf{v}(x, z, t) = (v_x, v_z) \), \( z = 0 \) is the position of the plate and \( z = h \) correspond to the fluid at rest.

* Corresponding author.
E-mail address: mederic.argentina@unice.fr (M. Argentina).
where $\varepsilon = h/L \ll 1$ is the ratio between the characteristic lengths of the problem and $Re = \Omega h^2/\nu$ is the Reynolds number. The kinematic and vertical dynamic boundary conditions are

$$v_z|_{z=\zeta} = \partial_t \zeta + v_x|_{z=\zeta} \partial_x \zeta,$$

(3)

$$\partial_z v_x|_{z=\zeta} + \zeta^2 \left( \partial_z^2 v_x|_{z=\zeta} - 2 \partial_x \zeta \partial_z v_x|_{z=\zeta} + \partial_x v_x|_{z=\zeta} (\partial_x \zeta)^2 \right) + O(\varepsilon^4) = 0,$$

(4)

and for the tangential component,

$$\eta|_{z=\zeta} + 2 \zeta \partial_x (v_x|_{z=\zeta}) + O(\varepsilon^4) = G(t) [\zeta - 1] + B \kappa,$$

(5)

where $G(t) = g(t) h^3 / \nu \Omega^2$, $B = \tau h^3 / \eta \Omega^4$, $\tau$ is the surface tension, $\eta = \nu \nu$ is the dynamic viscosity and $\kappa$ is the curvature of the surface.

In order to develop the nonlinear equation for the interfacial disturbance $\xi(x,t)$ and the mass flux $q(x,t)$, we need further approximations. We assume $Re \ll 1$ and we neglect $O(\varepsilon^4, \varepsilon^2 Re, Re^2, \ldots)$ terms. Since the fluid layer is thin, we perform a Taylor expansion with the variable $z$, where incompressibility and the no-slip boundary condition on the plate have been used:

$$v_x(x, z, t) = \sum_{n=0}^{\infty} \frac{v_n(x, t)}{(n+1)!} z^{n+1},$$

$$v_z(x, z, t) = - \sum_{n=0}^{\infty} \frac{\partial_z v_n(x, t)}{(n+2)!} z^{n+2},$$

$$\eta(x, z, t) = \sum_{n=0}^{\infty} \eta_n(x, t) \frac{z^n}{n!}.$$

(6)

By replacing these expansions into kinematic condition (3) we find a continuity equation and the related mass flux $q(x,t)$

$$\partial_t \zeta + \partial_x q = 0, \quad q(x,t) = \int_0^1 v_x(x, z, t) dz.$$

(7)

Replacing Eq. (6) into Eq. (2) and solving for each order we get at order $O(\varepsilon^4)$, the evolution equation for $q$, the remaining functions are neglected at order $O(\varepsilon^4, \varepsilon^2 Re, Re^2, \ldots)$.

$$\frac{6}{5} Re \zeta^2 \partial_z q + 3 q + \zeta^3 \left[ G(t) - B \zeta^2 \right] \partial_z \zeta - \frac{6 Re}{35} \left[ 9q^2 \partial_x \zeta - 17 \zeta \partial_x q \right] - \frac{6 \zeta^2}{5} \left[ 9 \frac{\partial_x q}{\zeta} - 6 \frac{\partial_x \zeta}{\zeta} - \frac{4 (\partial_x \zeta)^2}{\zeta^2} q - \frac{9}{2} \frac{\partial_x q}{\zeta^2} \frac{\partial_x \zeta}{\zeta} \right] = 0.$$

(8)

In the extremely viscous ($Re \rightarrow 0$) and shallow water ($\varepsilon^2 \rightarrow 0$) limits, Eqs. (7 and 8) yield to the Reynolds equation, which was found

![Fig. 1. 2D view of the experiment. The free surface is obtained from Eqs. (7 and 8), the parameters are taken from Fig. 2.](image)

![Fig. 2. (a) Spatio-temporal evolution of the surface in the asymptotic regime (b) surface's profile (blue line) far away from the instability threshold and mass flux profile (red line). The parameters used are $Re = 0.6, \varepsilon = 0.5, G = 25.0, \tau = 3.8, B = 0.7$ dx = 0.25, dt = 0.01.](image)
previously in [18,19] and does not show any instability in the non-
surfactant case.

\[ A = A_x G(t) - A^2 x_{hi} A^2 x_n / C_{16}/C_{17} \]  
\[ (9) \]

In the general case, the linearization of Eq. (8) together with
Eq. (7) give the Mathieu equation predicted in [12] for corresponding
limit (\( kh \ll 1 \))

\[ 6 Re A^2 t_n + 3 A_t^2 \xi - [G(t) - B A^2] \xi + O(e^2, \epsilon^2, Re^2, ...) = 0. \]

The generalization of the three dimensional case is straightforward,
and its spatio-temporal dynamics together with a two commensurate frequencies forcing will be the subject of another
work [22].

3. Results

Numerical solutions of our system Eqs. (7) and (8) are computed
using a staggered mesh [20]. The derivatives are approximated with
finite differences, and the temporal scheme is a Runge Kutta at fourth
order [21]. Numerical simulations still converge for \( Re = 1 \).

A single frequency forcing has been chosen \( (G(t) = G_0 + T \cos \Omega t) \).
In this case, as \( \Gamma \) becomes high enough \( (\Gamma > \Gamma^*) \), an instability develops
with a given wavenumber predicted with the linear theory [11,12].
After a transient regime, the amplitude of the surface deformation
saturates to a finite value, due to the non-linear terms as pictured in
Fig. 2(a). In Fig. 2(b), we show the \( q \) and \( \xi \) spatial profile for a given
time. At very low Reynolds number, the temporal dynamics are strongly
non-linear. In our parameter regime, the bifurcation appears super-
critical. Nevertheless, as the \( \Gamma \) increases, bistability appears. In Fig. 3,
we present the asymptotic wavelength as \( \Gamma \) increases from \( \Gamma^* \), and we
observe that, in the neighborhood of the bifurcation, there exists
bistability between two patterns with two different wavelengths
corresponding to subharmonic responses of the fluid.

\[ Fig. 3. (a) Main wavelength of the surface waves for increasing \( \Gamma \) (blue curve) and for decreasing \( \Gamma \) (red curve). Hysteresis is found at the transition between \( k_1 = 1.131 \) and \( k_2 = 1.382. \) No hysteresis is found at the threshold acceleration \( \Gamma^* \). (b) and (c) spatiotemporal subharmonic state for \( \Gamma = 4.0 \) with an initial condition \( \xi(x) = 1 + 10^{-8} \cos(k_{x}x) \) and \( \xi(x) = 1 + 10^{-8} \cos(k_{x}x) \). Bistability is found at the hysteresis region. The parameters value are the same as in Fig. 2. \]

\[ Fig. 4. Velocity field of the vibrated fluid film using Eqs. (7) and (8). The interface is represented by the red curve. The amplitude of the velocity is plotted in color, the white area correspond to the highest value of the norm of the velocity, whereas the dark area the lowest value. The values of the parameters are \( Re = 0.5, \epsilon = 0.5, G = 32.41, B = 74.07, \Gamma = 3, dx = 0.125, dt = 0.001. \)

Our approach also permits to evaluate the velocity field in the fluid,
as pictured in Fig. 4.

Acknowledgement

N. O. Rojas thanks the financial support of CONICYT. Simulations
have been done using the XDIm Software developed by M. Monticelli
and P. Coullet from the University of Nice.
References